XVI. *On the Communication of Vibration from a Vibrating Body to a surrounding Gas.*

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In the first volume of the Transactions of the Cambridge Philosophical Society is a short paper by Professor John Leslie, "On Sounds excited in Hydrogen Gas," in which the author mentions some remarkable experiments indicating the singular incapacity of hydrogen for becoming the vehicle of the transmission of sound when a bell is struck in that gas, either pure or mixed with air. With reference to the most striking of his experiments the author observes (p. 267), "The most remarkable fact is, that the admixture of hydrogen gas with atmospheric air has a predominant influence in blunting or stifling sound. If one half of the volume of atmospheric air be extracted [from the receiver of the air-pump], and hydrogen gas be admitted to fill the vacant space, the sound will now become scarcely audible."

No definite explanation of the results is given, but with reference to the feebleness of sound in hydrogen the author observes, "These facts, I think, depend partly on the tenuity of hydrogen gas, and partly on the rapidity with which the pulsations of sound are conveyed through this very elastic medium;" and he states that, according to his view, he "should expect the intensity of sound to be diminished 100 times, or in the compound ratio of its tenuity and of the square of the velocity with which it conveys the vibratory impressions." With reference to the effect of the admixture of hydrogen with air he says, "When hydrogen gas is mixed with common air, it probably does not intimately combine, but dissipates the pulsatory impressions before the sound is vigorously formed."

In referring to Leslie's experiment in which a half-exhausted receiver is filled up with hydrogen, Sir John Herschel suggests a possible explanation founded on Dalton's hypothesis that every gas acts as a vacuum towards every other*. According to this view there is a constant tendency for sound-waves to be propagated with different velocities in the air and hydrogen of which the mixture consists, but this tendency is constantly checked by the resistance which one gas opposes to the passage of another, calling into play something analogous to internal friction, whereby the sound-vibration though at first produced is rapidly stifled. Air itself indeed is a mixture; but the velocities of propagation of sound in nitrogen and oxygen are so nearly equal that the effect is supposed not to be sensible in this case.

* Encyclopædia Metropolitana, vol. iv. Art. Sound, § 108.

This explanation never satisfied me, believing, as I always have done, for reasons which it would take too long here to explain, that for purely hydrodynamical phenomena (such as those of sound) an intimate mixture of gases was equivalent to a single homogeneous medium. I had some idea of repeating the experiment, thinking that possibly LESLIE might not have allowed sufficient time for the gases to be perfectly mixed, though that did not appear likely, when another explanation occurred to me, which immediately struck me as being in all probability the true one.

In reading some years ago an investigation of Mr. EARNSHAW'S, in which a certain result relating to the propagation of sound in a straight tube was expressed in terms among other things of the velocity of propagation, the idea occurred to me that the high velocity of propagation of sound in hydrogen would account for the result of LESLIE'S experiment, though in a manner altogether different from anything relating to the propagation of sound in one dimension only.

Suppose a person to move his hand to and fro through a small space. The motion which is occasioned in the air is almost exactly the same as it would have been if the air had been an incompressible fluid. There is a mere local reciprocating motion, in which the air immediately in front is pushed forwards, and that immediately behind impelled after the moving body, while in the anterior space generally the air recedes from the encroachment of the moving body, and in the posterior space generally flows in from all sides, to supply the vacuum which tends to be created; so that in lateral directions the motion of the fluid is backwards, a portion of the excess of fluid in the front going to supply the deficiency behind. Now conceive the periodic time of the motion to be continually diminished. Gradually the alternation of movement becomes too rapid to permit of the full establishment of the merely local reciprocating flow; the air is sensibly compressed and rarefied, and a sensible sound-wave (or wave of the same nature, in case the periodic time be beyond the limits suitable to hearing) is propagated to a distance. The same takes place in any gas; and the more rapid be the propagation of condensations and rarefactions in the gas, the more nearly will it approach, in relation to the motions we have under consideration, to the condition of an incompressible fluid; the more nearly will the conditions of the displacement of the gas at the surface of the solid be satisfied by a merely local reciprocating flow.

This explanation when once it suggested itself seemed so simple and obvious that I could not doubt that it afforded the true mode of accounting for the phenomenon. It remained only to test the accuracy of the assigned cause by actual numerical calculation in some case or cases sufficiently simple to permit of precise analytical determination. The result of calculations of the kind applied to a sphere proved that the assigned cause was abundantly sufficient to account for the observed result. I have not hitherto published these results; but as the phenomenon has not to my knowledge been satisfactorily explained by others, I venture to hope that the explanation I have to offer, simple as it is in principle, may not be unworthy of the notice of the Royal Society.

For the purpose of exact analytical investigation I have taken the two cases of a vibrating sphere and a long vibrating cylinder, the motion of the fluid in the latter case being supposed to be in two dimensions. The sphere is chosen as the best representative of a bell, among the few geometrical forms of body for which the problem can be solved. The cylinder is chosen as the representative of a vibrating string. In the case of the sphere the problem is identical with that solved by POISSON in his memoir "Sur les movements simultanés d'un pendule et de l'air environnant"*, but the solution is discussed with a totally different object in view, and is obtained from the beginning, to avoid the needless complexity introduced by taking account of the initial circumstances, instead of supposing the motion already going on.

A. *Solution of the Problem in the case of a Vibrating Sphere.*

Suppose an elastic solid, spherical externally in its undisturbed position, to vibrate in the manner of a bell, the amplitude of vibration being very small. Suppose it surrounded by a homogeneous gas, which is at rest except in so far as it is set in motion by the sphere; and let it be required to determine the motion of the gas in terms of that of the sphere supposed given. We may evidently for the purposes of the present problem suppose the gas not to be subject to the action of external forces.

Let the gas be referred to the rectangular axes of x, y, z, and let u, v, w be the components of the velocity. Since the gas is at rest except as to the disturbance communicated to it from the sphere, u, v, w are by a well-known theorem the partial differential coefficients with respect to x, y, z of a function ϕ of the coordinates; and if a^2 be the constant expressing the ratio of the small variations of pressure to the corresponding small variations of density, we must have

$$\frac{d^2\phi}{dt^2}=a^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right); \quad \ldots \quad (1)$$

and if s be the small condensation,

$$s=-\frac{1}{a^2}\frac{d\phi}{dt}.$$

As we have to deal with a sphere, it will be convenient to refer the gas to polar coordinates r, θ, ω, the origin being in the centre. In terms of these coordinates, equation (1) becomes

$$\frac{d^2\phi}{dt^2}=a^2\left\{\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{d^2\phi}{d\omega^2}\right\}; \quad \ldots \quad (2)$$

and if u', v', w' be the components of the velocity along the radius vector and in two directions perpendicular to the radius vector, the first in and the second perpendicular to the plane in which θ is measured,

$$u'=\frac{d\phi}{dr}, \quad v'=\frac{1}{r}\frac{d\phi}{d\theta}, \quad w'=\frac{1}{r\sin\theta}\frac{d\phi}{d\omega}. \quad \ldots \quad (3)$$

Let c be the radius of the sphere, and V the velocity of any point of its surface

* Mémoires de l'Académie des Sciences, t. xi. p. 521.

resolved in a direction normal to the surface, V being a given function of t, θ, and ω; then we must have

$$\frac{d\varphi}{dr}=\mathrm{V}, \text{ when } r=c. \qquad (4)$$

Another condition, arising from what takes place at a great distance from the sphere, will be considered presently.

The sphere vibrating under the action of its elastic forces, its motion will be periodic, expressed so far as the time is concerned partly by the sine and partly by the cosine of an angle proportional to the time, suppose mat. Actually the vibrations will slowly die away, in consequence partly of the imperfect elasticity of the sphere, partly of communication of motion to the gas, but for our present purpose this need not be taken into account. Moreover there will in general be a series of periodic disturbances coexisting, corresponding to different periodic times, but these may be considered separately. We might therefore assume

$$\mathrm{V}=\mathrm{U}\sin mat+\mathrm{U}'\cos mat,$$

but it will materially shorten the investigation to employ an imaginary exponential instead of circular functions. If we take

$$\mathrm{V}=\mathrm{U}e^{imat}, \qquad (5)$$

where i is an abbreviation for $\sqrt{-1}$, and determine φ by the conditions of the problem, the real and imaginary parts of φ and V must satisfy all those conditions separately; and therefore we may take the real parts alone, or the coefficients of i or $\sqrt{-1}$ in the imaginary parts, or any linear combination of these even after having changed the arbitrary constants which enter into the expression of the motion of the sphere, as the solution of the problem, according to the way in which we conceive the given quantity V expressed.

The function φ will be periodic in a similar manner to V, so that we may take

$$\varphi=\psi e^{imat}. \qquad (6)$$

As regards the periodicity merely, we might have had a term involving e^{-imat} as well as that written above; but it will be readily seen that in order to satisfy the equation of condition (4) the sign of the index of the exponential in φ must be the same as in V.

On substituting in (2) the expression for φ given by (6), the factor involving t will divide out, and we shall get for the determination of ψ a partial differential equation free from t. Now ψ may be expanded in a series of LAPLACE'S Functions so that

$$\psi=\psi_0+\psi_1+\psi_2+\ldots \qquad (7)$$

Substituting in the differential equation just mentioned, taking account of the fundamental equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_n}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2\psi_n}{d\omega^2}=-n(n+1)\psi_n,$$

and equating to zero the sum of the LAPLACE'S Functions of the same order, we find

$$\frac{d^2\psi_n}{dr^2}+\frac{2}{r}\frac{d\psi_n}{dr}-\frac{n(n+1)}{r^2}\psi_n+m^2\psi_n=0.$$

This equation belongs to a known integrable form. The integral is

$$r\psi_n=u_n e^{-imr}\left\{1+\frac{n(n+1)}{2\,.\,imr}+\frac{(n-1)n(n+1)(n+2)}{2\,.\,4(imr)^2}+\ldots\right\}$$
$$+u'_n e^{imr}\left\{1-\frac{n(n+1)}{2\,.\,imr}+\frac{(n-1)n(n+1)(n+2)}{2\,.\,4(imr)^2}-\ldots\right\},$$

u_n and u'_n being evidently LAPLACE'S Functions of the order n, since that is the case with ψ_n.

It will be convenient to take next the condition which has to be satisfied at a great distance from the sphere. When r is very large the series within braces may be reduced to their first terms 1, and we shall have

$$r\phi=e^{im(at-r)}\Sigma u_n+e^{im(at+r)}\Sigma u'_n.$$

The first of these terms denotes a disturbance travelling outwards from the centre, the second a disturbance travelling towards the centre, the amplitude of vibration in both cases, for a given phase, varying inversely as the distance from the centre. In the problem before us there is no disturbance travelling towards the centre, and therefore $\Sigma u'_n=0$, which requires that each function u'_n should separately be equal to zero. We have therefore simply

$$r\psi_n=u_n e^{-imr}\left\{1+\frac{n(n+1)}{2.imr}+\frac{(n-1)\ldots(n+2)}{2.4(imr)^2}+\ldots+\frac{1.2.3\ldots 2n}{2.4.6\ldots 2n(imr)^n}\right\}, \quad . \quad . \quad . \quad (8)$$

or, if we choose to reverse the series,

$$r\psi_n=u_n e^{-imr}\frac{1.3.5\ldots(2n-1)}{(imr)^n}\left\{1+\frac{2n}{1.2n}imr+\frac{(2n-2)2n}{1.2(2n-1)2n}(imr)^2\ldots+\frac{2.4.6\ldots 2n}{1.2.3\ldots 2n}(imr)^n\right\}. \quad (9)$$

Putting for shortness $f_n(r)$ for the multiplier of $u_n e^{-imr}$ on the right-hand member of (8) or (9), we shall have

$$\phi=\frac{1}{r}e^{im(at-r)}\Sigma u_n f_n(r).$$

It remains to satisfy the equation of condition (4). Put for shortness

$$\frac{d}{dr}\left\{\frac{1}{r}e^{-imr}f_n(r)\right\}=-\frac{1}{r^2}e^{-imr}\mathrm{F}_n(r),$$

so that

$$\mathrm{F}_n(r)=(1+imr)f_n(r)-rf'_n(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and suppose U expanded in a series of LAPLACE'S Functions,

$$\mathrm{U}_0+\mathrm{U}_1+\mathrm{U}_2+\ldots;$$

then substituting and equating the functions of the same order on the two sides of the equation, we have

$$\mathrm{U}_n=-\frac{1}{c^2}e^{-imc}\mathrm{F}_n(c)u_n,$$

and therefore

$$\phi=-\frac{c^2}{r}e^{im(at-r+c)}\Sigma\frac{\mathrm{U}_n}{\mathrm{F}_n(c)}f_n(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This expression contains the solution of the problem, and it remains only to discuss it.

At a great distance from the sphere the function $f_n(r)$ becomes ultimately equal to 1, and we have

$$\phi=-\frac{c^2}{r}e^{im(at-r+c)}\Sigma\frac{U_n}{F_n(c)} \quad . \quad . \quad . \quad . \quad (12)$$

It appears from (3) that the component of the velocity along the radius vector is of the order r^{-1}, and that in any direction perpendicular to the radius vector of the order r^{-2}, so that the lateral motion may be disregarded except in the neighbourhood of the sphere.

In order to examine the influence of the lateral motion in the neighbourhood of the sphere, let us compare the actual disturbance at a great distance with what it would have been if all lateral motion had been prevented, suppose by infinitely thin conical partitions dividing the fluid into elementary canals, each bounded by a conical surface having its vertex at the centre.

On this supposition the motion in any canal would evidently be the same as it would be in all directions if the sphere vibrated by contraction and expansion of the surface, the same all round, and such that the normal velocity of the surface was the same as it is at the particular point at which the canal in question abuts on the surface. Now if U were constant the expansion of U would be reduced to its first term U_0, and seeing that $f_0(r)=1$ we should have from (11)

$$\phi=-\frac{c^2}{r}e^{im(at-r+c)}\frac{U_0}{F_0(c)}.$$

This expression will apply to any particular canal if we take U_0 to denote the normal velocity at the sphere's surface for that particular canal; and therefore to obtain an expression applicable at once to all the canals we have merely to write U for U_0. To facilitate a comparison with (11) and (12) I shall, however, write ΣU_n for U. We have then

$$\phi=-\frac{c^2}{r}e^{im(at-r+c)}\frac{\Sigma U_n}{F_0(c)} \quad . \quad . \quad . \quad . \quad (13)$$

It must be remembered that this is merely an expression applicable at once to all the canals, the motion in each of which takes place wholly along the radius vector, and accordingly the expression is not to be differentiated with respect to θ or ω with the view of applying the formulæ (3).

On comparing (13) with the expression for the function ϕ in the actual motion at a great distance from the sphere (12), we see that the two are identical with the exception that U_n is divided by two different constants, namely $F_0(c)$ in the former case and $F_n(c)$ in the latter. The same will be true of the leading terms (or those of the order r^{-1}) in the expressions for the condensation and velocity*. Hence if the mode of vibration of

* Of course it would be true if the *complete* differential coefficients with respect to r of the right-hand members of (12) and (13) were taken, but then the former does not give the velocity u' except as to its leading term, since (12) has been deduced from the exact expression (11) by reducing $f_n(r)$ to its first term 1; nor again is it true, except as to terms of the order r^{-1}, of the actual motion of the unimpeded fluid that the whole velocity is in the direction of the radius vector.

the sphere is such that the normal velocity of its surface is expressed by a LAPLACE'S Function of any one order, the disturbance at a great distance from the sphere will vary from one direction to another according to the same law as if lateral motion had been prevented, the amplitude of excursion at a given distance from the centre varying in both cases as the amplitude of excursion, in a normal direction, of the surface of the sphere itself. The only difference is that expressed by the symbolic ratio $F_n(c):F_0(c)$. If we suppose $F_n(c)$ reduced to the form $\mu_n(\cos\alpha_n+\sqrt{-1}\sin\alpha_n)$, the amplitude of vibration in the actual case will be to that in the supposed case as μ_0 to μ_n, and the phases in the two cases will differ by $\alpha_0-\alpha_n$.

If the normal velocity of the surface of the sphere be not expressible by a single LAPLACE'S Function, but only by a series, finite or infinite, of such functions, the disturbance at a given great distance from the centre will no longer vary from one direction to another according to the same law as the normal velocity of the surface of the sphere, since the modulus μ_n and likewise the amplitude α_n of the imaginary quantity $F_n(c)$ vary with the order of the function.

Let us now suppose the disturbance expressed by a LAPLACE'S Function of some one order, and seek the numerical value of the alteration of intensity at a distance, produced by the lateral motion which actually exists.

The intensity will be measured by the *vis viva* produced in a given time, and consequently will vary as the density multiplied by the velocity of propagation multiplied by the square of the amplitude of vibration. It is the last factor alone that is different from what it would have been if there had been no lateral motion. The amplitude is altered in the proportion of μ_0 to μ_n, so that if

$$\frac{\mu_n^2}{\mu_0^2}=I_n,$$

I_n is the quantity by which the intensity which would have existed if the fluid had been hindered from lateral motion has to be divided.

For the first five orders the values of the function $F_n(c)$ are as follows:—

$$F_0(c)=imc+1,$$

$$F_1(c)=imc+2+\frac{2}{imc},$$

$$F_2(c)=imc+4+\frac{9}{imc}+\frac{9}{(imc)^2},$$

$$F_3(c)=imc+7+\frac{27}{imc}+\frac{60}{(imc)^2}+\frac{60}{(imc)^3},$$

$$F_4(c)=imc+11+\frac{65}{imc}+\frac{240}{(imc)^2}+\frac{525}{(imc)^3}+\frac{525}{(imc)^4}.$$

If λ be the length of the sound-wave corresponding to the period of the vibration, $m=\frac{2\pi}{\lambda}$, so that mc is the ratio of the circumference of the sphere to the length of a wave. If we suppose the gas to be air and λ to be 2 feet, which would correspond to about 550 vibrations in a second, and the circumference $2\pi c$ to be 1 foot (a size and pitch which would correspond with the case of a common house bell), we shall have $mc=\frac{1}{2}$. The following Table gives the values of the square of the modulus and of the ratio I_n for the functions $F_n(c)$ of the first five orders, for each of the values 4, 2, 1, $\frac{1}{2}$, and $\frac{1}{4}$ of mc. It will presently appear why the Table has been extended further in the direction of values greater than $\frac{1}{2}$ than it has in the opposite direction. Five significant figures at least are retained.

mc.	$n=0$.	$n=1$.	$n=2$.	$n=3$.	$n=4$.	
4	17	16·25	14·879	13·848	20·177	Values of μ_n^2.
2	5	5	9·3125	80	1495·8	
1	2	5	89	3965	300137	
0·5	1·25	16·25	1330·2	236191	72086371	
0·25	1·0625	64·062	20878	14837899	18160×10^6	
4	1	0·95588	0·87523	0·81459	1·1869	Values of I_n.
2	1	1	1·8625	16	299·16	
1	1	2·5	44·5	1982·5	150068	
0·5	1	13	1064·2	188953	57669097	
0·25	1	60·294	19650	13965×10^3	17092×10^6	

When $mc=\infty$ we get from the analytical expressions $I_n=1$. We see from the Table that when mc is somewhat large I_n is liable to be *a little* less than 1, and consequently the sound to be *a little* more intense than if lateral motion had been prevented. The possibility of this is explained by considering that the waves of condensation spreading from those compartments of the sphere which at a given moment are vibrating positively, *i. e.* outwards, after the lapse of a half period may have spread over the neighbouring compartments, which are now in their turn vibrating positively, so that these latter compartments in their outward motion work against a somewhat greater pressure than if each compartment had opposite to it only the vibration of the gas which it had itself occasioned; and the same explanation applies *mutatis mutandis* to the waves of rarefaction. However, the increase of sound thus occasioned by the existence of lateral motion is but small in any case, whereas when mc is somewhat small I_n increases enormously, and the sound becomes a mere nothing compared with what it would have been had lateral motion been prevented.

The higher be the order of the function, the greater will be the number of compartments, alternately positive and negative as to their mode of vibration at a given moment, into which the surface of the sphere will be divided. We see from the Table that for a given periodic time as well as radius the value of I_n becomes considerable when n is somewhat high. However practically vibrations of this kind are produced when the elastic sphere executes, not its principal, but one of its subordinate vibrations, the pitch corresponding to which rises with the order of the vibration, so that m increases with that order. It was for this reason that the Table was extended from

$mc=0{\cdot}5$ further in the direction of high pitch than low pitch, namely, to three octaves higher and only one octave lower.

When the sphere vibrates symmetrically about the centre, *i. e.* so that any two opposite points of the surface are at a given moment moving with equal velocities in opposite directions, or more generally when the mode of vibration is such that there is no change of position of the centre of gravity of the volume, there is no term of the order 1. For a sphere vibrating in the manner of a bell the principal vibration is that expressed by a term of the order 2, to which I shall now more particularly attend.

Putting, for shortness, $m^2c^2=q$, we have

$$\mu_0^2=q+1,\ \mu_2^2=(q^{\frac{1}{2}}-9q^{-\frac{1}{2}})^2+\left(4-\frac{9}{q}\right)^2=q-2+\frac{9}{q}+\frac{81}{q^2},$$
$$\mathrm{I}_2=\frac{q^3-2q^2+9q+81}{q^2(q+1)}.$$

The minimum value of I_2 is determined by

$$q^3-6q^2-84q-54=0,$$

giving approximately

$$q=12{\cdot}859,\ mc=3{\cdot}586,\ \mu_0^2=13{\cdot}859,\ \mu_2^2=12{\cdot}049,\ \mathrm{I}_2={\cdot}86941;$$

so that the utmost increase of sound produced by lateral motion amounts to about 15 per cent.

I come now more particularly to LESLIE'S experiments. Nothing is stated as to the form, size, or pitch of his bell; and even if these had been accurately described, there would have been a good deal of guesswork in fixing on the size of the sphere which should be considered the best representative of the bell. Hence all we can do is to choose such values for m and c as are comparable with the probable conditions of the experiment.

I possess a bell, belonging to an old bell-in-air apparatus, which may probably be somewhat similar to that used by LESLIE. It is nearly hemispherical, the diameter is 1·96 inch, and the pitch an octave above the middle C of a piano. Taking the number of vibrations 1056 per second, and the velocity of sound in air 1100 feet per second, we have $\lambda=12{\cdot}5$ inches. To represent the bell by a sphere of the same radius would be very greatly to underrate the influence of local circulation, since near the mouth the gas has but a little way to get round from the outside to the inside, or the reverse. To represent it by a sphere of half the radius would still apparently be to underrate the effect. Nevertheless for the sake of rather underestimating than exaggerating the influence of the cause here investigated, I will make these two suppositions successively, giving respectively $c={\cdot}98$ and $c={\cdot}49$, $mc={\cdot}4926$, and $mc={\cdot}2463$ for air.

If it were not for lateral motion the intensity would vary from gas to gas in the proportion of the density into the velocity of propagation, and therefore as the pressure into the square root of the density under a standard pressure, if we take the factor depending on the development of heat as sensibly the same for the gases and gaseous mixtures with which we have to deal. In the following Table the first column gives the gas, the second the pressure p, in atmospheres, the third the density D under the pressure p, referred to the density of air at the atmospheric pressure as unity, the fourth, Q_r, what would have been the intensity had the motion been wholly radial, referred to the intensity in air at atmospheric pressure as unity, or, in other words, a quantity varying as $p \times$ (the density at pressure $1)^{\frac{1}{2}}$. Then follow the values of q, I_2, and Q, the last being the actual intensity referring to air as before.

Gas.	p.	D.	Q_r.	$c = \cdot 98$.			$c = \cdot 49$.		
				q.	I_2.	Q.	q.	I_2.	Q.
Air	1	1	1	·2427	1136	1	·06067	20890	1
Hydrogen	1	·0690	·2627	·01674	284700	·001048	·004186	4604000	·001191
Air, rarefied	·01	·01	·01	·2427	1136	·01	·06067	20890	·01
The same filled with H	1	·0783	·2798	·01900	220600	·001440	·004751	3572000	·001637
Air of same density	·0783	·0783	·0783	·2427	1136	·0783	·06067	20890	·0783
Air rarefied $\frac{1}{2}$	·5	·5	·5	·2427	1136	·5	·06067	20890	·5
The same filled with H	1	·5345	·7311	·1297	4322	·1921	·0324	74890	·2039

An inspection of the numbers contained in the columns headed Q will show that the cause here investigated is amply sufficient to account for the facts mentioned by LESLIE.

It may be noticed that while q is 4 times smaller, and I_2 is 16 or 18 times larger, for $c = \cdot 49$ than for $c = \cdot 98$, there is no great difference in the values of Q in the two cases for hydrogen and mixtures of hydrogen and air in given proportions. This arises from the circumstance that q is sufficiently small to make the last terms in μ_0^2 and μ_2^2, namely, 1 and $81q^{-2}$, the most important, so that I_n does does not greatly differ from $81q^{-2}$. If this result had been exact instead of approximate, the intensity in different gases, supposed for simplicity to be at a common pressure, would have varied as $D^{\frac{5}{2}}$; and it will be found that for the cases in which $p=1$ the values of Q in the above Table, especially those in the last column, do not greatly deviate from this proportion. But the simplicity of this result depends on two things. First, the vibration must be expressed by a LAPLACE'S function of the order 2; for a different order the power of D would have been different; and this is just one of the points respecting which we cannot infer what would be true of a bell of the ordinary shape from what we have proved for a sphere. Secondly, the radius must be sufficiently small, or the pitch sufficiently low, to make q small; at the other extremity of the scale, in which c is supposed to be very large, or λ very small, Q varies nearly as $D^{\frac{1}{2}}$ instead of $D^{\frac{5}{2}}$, whatever be the order of the LAPLACE'S function. Hence no simple relation can be expected between the numbers furnished by experiment and the numerical constants of the gas in such experiments as those of M. PEROLLE*, in which the same bell was rung in succession in different gases.

B. *Solution of the Problem in the case of a Vibrating Cylinder.*

I will here suppose the motion to be in two dimensions only. In the case of a vibrating string, which I have mainly in view, it is true that the amplitude of excursion

* Mémoires de l'Académie des Sciences de Turin, t. iii. (1786–7); Mém. des Correspondans, p. 1.

of the string varies sensibly on proceeding even a moderate distance along it, and that the propagation of the sound-wave produced by no means takes place in two dimensions only. But the question how far a sound-wave is produced at all, and how far the displacement of the gas by the cylinder merely produces a local motion to and fro, is decided by what takes place in the immediate neighbourhood of the string, such as within a distance of a few diameters; and though the sound-wave, when once produced, in its subsequent progress diverges in three dimensions, the same takes place with the hypothetical sound-wave which would be produced if lateral motion were prevented, so that the comparison which it is the object of the investigation to institute is not affected thereby.

Assuming, then, the motion to be in two dimensions, and referring the fluid to polar coordinates, r, θ, r being measured from the axis of the undisturbed cylinder, we shall have for the fundamental equation derived from (1)

$$\frac{d^2\varphi}{dt^2}=a^2\left\{\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}+\frac{1}{r^2}\frac{d^2\varphi}{d\theta^2}\right\}; \quad \ldots \ldots \quad (14)$$

and if u', v' be the components of the velocity along and perpendicular to the radius vector,

$$u'=\frac{d\varphi}{dr}, \qquad v'=\frac{1}{r}\frac{d\varphi}{d\theta}.$$

If c be the radius of the cylinder, and V the normal component of the velocity of the surface of the cylinder, we must have

$$\frac{d\varphi}{dr}=\mathrm{V}, \text{ when } r=c.$$

As before, I will suppose the motion of the cylinder, and consequently of the fluid, to be regularly periodic, but instead of using circular functions directly I will employ the imaginary exponential e^{imat}, i denoting as before $\sqrt{-1}$, and will put accordingly $\mathrm{V}=e^{imat}\mathrm{U}$, U being a given function of θ, and $\varphi=\psi e^{imat}$. For a given value of r, ψ may by a known theorem be developed in a series of sines and cosines of θ and its multiples, and therefore for general values of r can be so developed, the coefficients being functions of r. If ψ_n be the coefficient of cos $n\theta$ or sin $n\theta$, we find

$$\frac{d^2\psi_n}{dr^2}+\frac{1}{r}\frac{d\psi_n}{dr}-\frac{n^2}{r^2}\psi_n+m^2\psi_n=0. \quad \ldots \ldots \quad (15)$$

If we suppose the normal velocity of the surface of the cylinder to vary in a given manner from one generating line to another, so that U is a given function of θ, we may expand U in a series of the form

$$\mathrm{U}=\mathrm{U}_0+\mathrm{U}_1\cos\theta+\mathrm{U}_2\cos 2\theta+\ldots$$
$$+\mathrm{U}'_1\sin\theta+\mathrm{U}'_2\sin 2\theta+\ldots$$

On applying now the equation of condition which has to be satisfied at the surface of the cylinder, we see that a term $\mathrm{U}_n\cos n\theta$ or $\mathrm{U}'_n\sin n\theta$ of the nth order in the expression for U will introduce a function ψ_n of the same order in the general expression for ϕ. Now the only case of interest relating to an infinite cylinder is that of a vibrating string, in which the cylinder moves as a whole. The vibration may be regarded as compounded of the vibrations in any two rectangular planes passing through the axis, the phases of the component vibrations, it may be, being different. These component vibrations may be treated separately, and thus it will suffice to suppose the vibration confined to one plane, which we may take to be that from which θ is measured. We shall accordingly have

$$U=U_1\cos\theta,$$

U_1 being a given constant, and the only function ψ_n which will appear in the general expression for ϕ will be that of the order 1. Besides this we shall have to investigate, for the sake of comparison, an ideal vibration in which the cylinder alternately contracts and expands in all directions alike, and for which accordingly U is a constant U_0. Hence the equation (15) need only be considered for the values 0 and 1 of n.

For general values of n the equation (15) is easily integrated in the form of infinite series according to ascending powers of r. The result is

$$\left.\begin{aligned}\psi_n=&Ar^n\left\{1-\frac{m^2r^2}{2(2+2n)}+\frac{m^4r^4}{2.4(2+2n)(4+2n)}-\dots\right\}\\ &+Br^{-n}\left\{1-\frac{m^2r^2}{2(2-2n)}+\frac{m^4r^4}{2.4(2-2n)(4-2n)}-\dots\right\}\end{aligned}\right\}\quad\dots\dots\quad(16)$$

When n is any integer the integral as it stands becomes illusory; but the complete integral, which in this case assumes a special form, is readily obtained as a limiting case of the complete integral for general values of n.

The series in (16) are convergent for any value of r however great, but they give us no information of what becomes of the functions for very large values of r.

When r is very large, the equation (15) becomes approximately

$$\frac{d^2\psi_n}{dr^2}+m^2\psi_n=0,$$

the integral of which is $\psi_n=Re^{-imr}+R'e^{imr}$, where R and R′ are constant. This suggests putting the complete integral of (15) under the same form, R and R′ being now functions of r, which, when r is large, vary but slowly, that is, remain nearly constant when r is altered by only a small multiple of λ. Assuming for R and R′ series of the form $Ar^\alpha+Br^\beta+Cr^\gamma\dots$, where α, β, $\gamma\dots$ are in decreasing order algebraically, and determining the indices and coefficients so as to satisfy (15), we get for another form of the complete integral

$$\left.\begin{aligned}\psi_n=&C(imr)^{-\frac{1}{2}}e^{-imr}\left\{1-\frac{1^2-4n^2}{1.8imr}+\frac{(1^2-4n^2)(3^2-4n^2)}{1.2(8imr)^2}-\frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1.2.3(8im)^3}+\dots\right\}\\ &+D(imr)^{-\frac{1}{2}}e^{imr}\left\{1+\frac{(1^2-4n^2)}{1.8imr}+\frac{(1^2-4n^2)(3^2-4n^2)}{1.2(8imr)^2}+\frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1.2.3(8imr)^3}+\dots\right\}\end{aligned}\right\}\ (17)$$

These series, though ultimately divergent, begin by converging rapidly when r is large, and may be employed with great advantage when r is large, if we confine ourselves to the converging part. Moreover we have at once D=0 as the condition to be satisfied at a great distance from the cylinder. If mc were large we might employ the second form of integral in satisfying the condition at the surface of the cylinder, and the problem would present no further difficulty. But practically in the case of vibrating strings mc is a very small fraction; the series (16) are rapidly convergent, and the series (17) cannot be employed. To complete the solution of the problem therefore it is essential to express the constants A and B in terms of C and D, or at any rate to find the relation between A and B imposed by the condition D=0.

This may be effected by means of the complete integral of (15) expressed in the form of a definite integral. For $n=0$ we know that

$$\psi_0=\int_0^{\frac{\pi}{2}}\{E+F\log(r\sin^2\zeta)\}\cos(mr\cos\zeta)d\zeta \quad . \quad . \quad . \quad . \quad . \quad (18)$$

is a third form of the integral of (15). It is not difficult to deduce from this the integral of (15) in a similar form for any integral value of n. Assuming

$$\psi_n=r^\alpha\int r^\beta\chi_n dr,$$

and substituting in (15), we have

$$r^{\alpha+\beta}\frac{d\chi_n}{dr}+(2\alpha+\beta+1)r^{\alpha+\beta-1}\chi_n+(\alpha^2-n^2)r^{\alpha-2}\int r^\beta\chi_n dr+m^2r^\alpha\int r^\beta\chi_n dr=0.$$

Assume

$$\alpha^2-n^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

divide the equation by r^α, differentiate with respect to r, and then divide by r^β. The result is

$$\frac{d^2\chi_n}{dr^2}+\frac{2\alpha+2\beta+1}{r}\frac{d\chi_n}{dr}+(2\alpha+\beta+1)(\beta-1)\frac{\chi_n}{r^2}+m^2\chi_n=0.$$

This equation will be of the same form as (15) provided,

$$\alpha+\beta=0,$$

which reduces the coefficient of the last term but one to $-(\alpha+1)^2$. In order that this coefficient may be increased we must choose the positive root of (19), namely n, which I will suppose positive. Hence

$$\psi_n=r^n\int r^{-n}\chi_n dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

gives

$$\frac{d^2\chi_n}{dr^2}+\frac{1}{r}\frac{d\chi_n}{dr}-\frac{(n+1)^2}{r^2}\chi_n+m^2\chi_n=0,$$

the same equation as that for the determination of ψ_{n+1}. Hence expressing χ_n in terms of ψ_n from (20), writing $n-1$ for n, and replacing χ_{n-1} by ψ_n, we have

$$\psi_n=r^{n-1}\frac{d}{dr}r^{-(n-1)}\psi_{n-1},$$

a formula of reduction which when n is integral serves to express ψ_n in terms of ψ_0. We have

$$\psi_n = r^n\left(\frac{1}{r}\frac{d}{dr}\right)^n \psi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

an equation which when applied to (18) gives the complete integral of (15) for integral values of n in the form of a definite integral.

Let us attend now more particularly to the case of $n=0$. The equation (16) is of the form $\psi_n = \mathrm{A}f(n) + \mathrm{B}f(-n)$, $f(n)$ containing r as well as n. Expanding by MACLAURIN'S Theorem, we have

$$\psi_n = (\mathrm{A}+\mathrm{B})f(0) + (\mathrm{A}-\mathrm{B})f'(0)n + (\mathrm{A}+\mathrm{B})f''(0)\frac{n^2}{1.2} + \ldots$$

Writing A for A+B, n^{-1}B for A−B, and then making n vanish, we have

$$\psi_0 = \mathrm{A}f(0) + \mathrm{B}f'(0),$$

or

$$\left.\begin{aligned}\psi_0 = (\mathrm{A}+\mathrm{B}\log r)\left(1 - \frac{m^2r^2}{2^2} + \frac{m^4r^4}{2^24^2} - \frac{m^6r^6}{2^24^26^2} + \ldots\right) \\ + \mathrm{B}\left(\frac{m^2r^2}{2^2}\mathrm{S}_1 - \frac{m^4r^4}{2^24^2}\mathrm{S}_2 + \frac{m^6r^6}{2^24^26^2}\mathrm{S}_3 - \ldots\right),\end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

where

$$\mathrm{S}_n = 1^{-1} + 2^{-1} + 3^{-1} \ldots + n^{-1}.$$

The integral in the form (17) assumes no peculiar shape when n is integral, and we have at once

$$\left.\begin{aligned}\psi_0 = \mathrm{C}(imr)^{-\frac{1}{2}}e^{-imr}\left\{1 - \frac{1^2}{1.8mr} + \frac{1^23^2}{1.2(8mr)^2} - \frac{1^23^25^2}{1.2.3(8imr)^3} + \ldots\right\} \\ + \mathrm{D}(imr)^{-\frac{1}{2}}e^{imr}\left\{1 + \frac{1^2}{1.8imr} + \frac{1^23^2}{1.2(8imr)^2} - \frac{1^23^25^2}{1.2.3(8imr)^3} + \ldots\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

I have explained at length the mode of dealing with such functions, and especially of connecting the arbitrary constants in the ascending and descending series, in two papers published in the Transactions of the Cambridge Philosophical Society*, in the second of which the connexion of the constants is worked out in this very example. To these I will refer, merely observing that while it is perfectly easy to connect A, B with E, F, the connexion of C, D with E, F involves some extremely curious points of analysis. The result of eliminating E, F between the two equations connecting A, B with E, F and the two connecting C, D with E, F is given, except as to notation, in the two equations (41) of my second paper. To render the notation identical with that of the former paper, it will be sufficient to write A−B log (im)+B log (imr) for A+B log r, and x for imr. The equations referred to may be simplified by the introduction of EULER'S constant γ, the value of which is ·57721566 &c., since it is known that

$$\pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2}) + \log 4 + \gamma = 0,$$

* "On the Numerical Calculation of a Class of Definite Integrals and Infinite Series," vol. ix. p. 166, and "On the Discontinuity of Arbitrary Constants which appear in Divergent Developments," vol. x. p. 105. A supplement to the latter paper has recently been read before the Cambridge Philosophical Society.

$\Gamma'(x)$ denoting the derivative of the function $\Gamma(n)$. Putting

$$A - B\log im = A',$$

we have by equations (41) of the second paper referred to

$$C = (2\pi)^{-\frac{1}{2}}\{iA' + [(\log 2 - \gamma)i - \pi]B\}, \quad \ldots \quad (24)$$

$$D = (2\pi)^{-\frac{1}{2}}\{A' + (\log 2 - \gamma)B\}, \quad \ldots \quad (25)$$

i being written for $\sqrt{-1}$. It is shown in that paper that these values of C, D hold good when the amplitude of the imaginary variable x lies between the limits 0 and π, or that of r (supposed to be imaginary) between the limits $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, but in crossing either of these limits one or other of the constants C, D is changed. In the investigation of the present paper r is of course real.

We have now

$$A' = A - B\log im = (\gamma - \log 2)B$$

for the relation between A and B arising from the condition that the motion is propagated outwards from the cylinder; and substituting in (22), we have for the value of ψ_0 subject to this condition

$$\left.\begin{aligned}\psi_0 = {} & B\left(\gamma + \log\frac{imr}{2}\right)\left(1 - \frac{m^2r^2}{2^2} + \frac{m^4r^4}{2^24^2} - \ldots\right) \\ & + B\left(\frac{m^2r^2}{2^2}S_1 - \frac{m^4r^4}{2^24^2}S_2 + \ldots\right);\end{aligned}\right\} \quad \ldots \quad (26)$$

or expressed by means of the descending series,

$$\psi_0 = -B\left(\frac{\pi}{2imr}\right)^{\frac{1}{2}} e^{-imr}\left\{1 - \frac{1^2}{1.8imr} + \frac{1^23^2}{1.2(8imr)^2} - \frac{1^23^25^2}{1.2.3(8imr)^3} + \ldots\right\} \quad \ldots \quad (27)$$

We have from (21)

$$\psi_1 = \frac{d\psi_0}{dr},$$

from which the complete integral of (15) for $n=1$ may be got from that for $n=0$. In the form (17) of the integral the parts arising from differentiation of the parts containing e^{-imr} and e^{imr} respectively will contain those same exponentials, and therefore the complete integral of (15) for $n=1$, subject to the condition that the part containing e^{imr} shall disappear, will be got by differentiating the complete integral for $n=0$ subject to that same condition. The *form* of the integral in the shape of a descending series is given at once by (17). Hence we get by differentiating (26) and (27), and at the same time changing the arbitrary constant by writing B_1m^{-1} for B,

$$\left.\begin{aligned}\psi_1 = {} & \frac{B_1}{mr}\left\{1 - \frac{m^2r^2}{2^2} + \frac{m^4r^4}{2^2.4^2} - \ldots\right\} \\ & - B_1\left(\gamma + \log\frac{imr}{2}\right)\left(\frac{mr}{2} - \frac{m^3r^3}{2^2.4} + \frac{m^5r^5}{2^2.4^2.6} - \ldots\right\} \\ & + B_1\left(\frac{mr}{2}S_1 - \frac{m^3r^3}{2^2.4}S_2 + \frac{m^5r^5}{2^2.4^2.6}S_3 - \ldots\right\}\end{aligned}\right\} \quad \ldots \quad (28)$$

$$\psi_1 = B_1\left(\frac{\pi i}{2mr}\right)^{\frac{1}{2}} e^{-imr}\left\{1 - \frac{-1.3}{1.8mr} + \frac{-1.1.3.5}{1.2(8mr)^2} - \frac{-1.1.3^2.5.7}{1.2.3(8mr)^3} + \ldots\right\} \quad \ldots \quad (29)$$

To determine the arbitrary constants B_1 and B, the first belonging to the actual motion, the second to the motion which would take place if the fluid were confined by an infinite number of planes passing through the axis, we must have, as before, for $r=c$,

$$\frac{d\psi_0}{dr}=U_1, \quad \frac{d\psi_1}{dr}=U_1,$$

whence

$$\left.\begin{aligned}\frac{cU_1}{B}&=1-\frac{m^2c^2}{2^2}+\frac{m^4c^4}{2^2.4^2}-\ldots-\left(\gamma+\log\frac{mc}{2}+i\frac{\pi}{2}\right)\left(\frac{m^2c^2}{2}-\frac{m^4c^4}{2^2.4}+\ldots\right)\\&\quad+\frac{m^2c^2}{2}S_1-\frac{m^4c^4}{2^2.4}S_2+\frac{m^6c^6}{2^2.4^2.6}S_3-\ldots\\&=F_0(mc)+i\frac{\pi}{2}f_0(mc), \text{ suppose;}\end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

$$\left.\begin{aligned}\frac{cU_1}{B_1}&=-\frac{1}{mc}-\frac{3mc}{2^2}+\frac{7m^3c^3}{2^2.4^2}-\frac{11m^5c^5}{2^2.4^2.6^2}+\ldots\\&\quad-\left(\gamma+\log\frac{mc}{2}+i\frac{\pi}{2}\right)\left(\frac{mc}{2}-\frac{3m^3c^3}{2^2.4}+\ldots\right)\\&\quad+\frac{mc}{2}S_1-\frac{3m^3c^3}{2^24}S_2+\frac{5m^5c^5}{2^2.4^2.6}S_3-\ldots\\&=\frac{1}{mc}\left\{F_1(mc)+i\frac{\pi}{2}f_1(mc)\right\}, \text{ suppose.}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

If I be the ratio of the intensities at a distance in the supposed and in the actual case, we see from (30) and (31) that I will be equal to the ratio of the squares of the moduli of B and B_1, and we shall therefore have

$$I=\frac{\{4F_1(mc)\}^2+\pi^2\{f_1(mc)\}^2}{m^2c^2[4\{F_0(mc)\}^2+\pi^2\{f_0(mc)\}^2]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

For a piano string corresponding to the middle C, c may be about ·02 inch, and λ is about 25 inches. This gives $mc=\cdot005027$. For such small values of mc I does not sensibly differ from $(mc)^{-2}$, which in the present case is 39571, so that the sound is nearly 40000 times weaker than it would have been if the motion of the particles of air had taken place in planes passing through the axis of the string. This shows the vital importance of sounding-boards in stringed instruments. Although the amplitude of vibration of the particles of the sounding-board is extremely small compared with that of the particles of the string, yet as it presents a broad surface to the air it is able to excite loud sonorous vibrations, whereas were the string supported in an absolutely rigid manner, the vibrations which it could excite directly in the air would be so small as to be almost or altogether inaudible.

I may here mention a phenomenon which fell under my notice, and which is readily explained by the principles laid down in this paper. As I was walking one windy day on a road near Cambridge, on the other side of which ran a line of telegraph, my attention was attracted by a peculiar sound of extremely high pitch, which seemed to come from the opposite side of the road. On going over to ascertain the cause, I found that it came directly through air from the telegraph wires. On standing near a telegraph post, the ordinary comparatively bass sound with which we are so familiar was heard, appearing to emanate from the post. On receding from the post the bass sound became feebler, and midway between two posts was quite inaudible. Nothing was then heard but the peculiar high-pitched sound, which appeared to emanate from the wires overhead. It had a peculiar metallic ring about it which the ear distinguished from the whistling of the wind in the twigs of a bush. Although the telegraph ran for miles, it was only at one spot that the peculiar sound was noticed, and even there only in certain states of the wind. The wires seemed to be less curved than usual at the place in question, from which it may be inferred that they were there subject to an unusually great tension.

The explanation of the phenomenon is easy after what precedes. The wires were thrown into vibration by the wind, and a number of different vibrations, having different periodic times, coexisted. As regards the vibrations of comparatively long period, the air around the wires behaved nearly like an incompressible fluid, and no sonorous vibrations of sensible amount were produced. These vibrations of the wires were, however, communicated to the posts, which being broad acted as sounding-boards, and thus sonorous vibrations of corresponding period were indirectly excited in the air. But as regards the vibrations of extremely short periodic time, the wires in spite of their narrowness were able by acting directly on the air to produce condensations and rarefactions of sensible amount.

The diameter of the telegraph wire was about ·166 inch; and if we take the C below the middle C of a piano for the representative of the pitch of the lower note, and a note five octaves higher for that of the higher, we have in the first case $\lambda=50$ inches nearly, and in the second $\lambda=50\times2^{-5}$, giving in the former case $mc=\cdot01043$, and in the latter $mc=\cdot3338$ The former of these values is so small that we may take $I=(mc)^{-2}$; in the latter case the formula (32) gives for I a value a little less than $(mc)^{-2}$. We find in the two cases $I=9192$ and $I=7\cdot202$ respectively, so that in the former case the sound is more than 9000 times feebler than that corresponding to the amplitude of vibration of the wire on the supposition of the absence of lateral motion, whereas in the latter case the actual intensity is nearly one-seventh of the full intensity corresponding to the amplitude.

The increase of sound produced by the stoppage of lateral motion may be prettily exhibited by a very simple experiment. Take a tuning-fork, and holding it in the fingers after it has been made to vibrate, place a sheet of paper or the blade of a broad knife with its edge parallel to the axis of the fork, and as near to the fork as conveniently may be without touching. If the plane of the obstacle coincide with either of the planes of symmetry of the fork, as represented in section at A or B, no effect is produced; but if it be placed in an intermediate position, such as C, the sound becomes much stronger.